EXEL'S CROSSED PRODUCT AND RELATIVE CUNTZ-PIMSNER ALGEBRAS

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Abstract. We consider Exel's new construction of a crossed product of a C^*-algebra A by an endomorphism α. We prove that this crossed product is universal for an appropriate family of covariant representations, and we show that it can be realised as a relative Cuntz-Pimsner algbera. We describe a necessary and sufficient condition for the canonical map from A into the crossed product to be injective, and present several examples to demonstrate the scope of this result. We also prove a gauge-invariant uniqueness theorem for the crossed product.

1. Introduction

If α is an endomorphism of a C^*-algebra A, we can form a new C^*-algebra called the crossed product of A by α. This was first done by Cuntz [2], and there are now several general theories [14, 17, 13], which have been applied in a number of settings [1, 9, 10].

In [3], Exel proposed a new definition for the crossed product of a unital C^*-algebra A by an endomorphism α. Exel's crossed product depends not only on A and α, but also on the choice of a *transfer operator*, which is a positive continuous linear map $L : A \to A$ such that $L(\alpha(a)b) = aL(b)$ for $a, b \in A$. This new theory generalises previous constructions where the endomorphism is injective and has hereditary range [13], and has applications in the study of classical irreversible dynamical systems [5].

In this paper, we re-examine Exel's crossed product, denoted $A \rtimes_{\alpha,L} \mathbb{N}$, and identify a family of representations for which $A \rtimes_{\alpha,L} \mathbb{N}$ is universal. We then show that $A \rtimes_{\alpha,L} \mathbb{N}$ can be realised as a relative Cuntz-Pimsner algebra as in [11, 6], and use known results for relative Cuntz-Pimsner algebras to study $A \rtimes_{\alpha,L} \mathbb{N}$. In particular, we identify conditions which ensure that the canonical map $A \to A \rtimes_{\alpha,L} \mathbb{N}$ is injective, thus answering a question raised by Exel in [3], and partially answered by him in [4].

We begin with a brief discussion of relative Cuntz-Pimsner algebras, and we state a lemma which we will use when considering the map $A \to A \rtimes_{\alpha,L} \mathbb{N}$. In §3 we discuss representations of Exel's crossed product. The main result in this section is the realization of $A \rtimes_{\alpha,L} \mathbb{N}$ as a relative Cuntz-Pimsner algebra.

In §4 we describe a necessary and sufficient condition on the transfer operator L for $A \to A \rtimes_{\alpha,L} \mathbb{N}$ to be injective. We also show that this condition simplifies when A is a commutative C^*-algbera, and give examples to illustrate that our results do significantly improve those of Exel. In §5 we use our realisation of $A \rtimes_{\alpha,L} \mathbb{N}$ as a relative Cuntz-Pimsner algebra and results of Katsura [8] and Muhly-Tomforde [12] to prove a gauge-invariant uniqueness theorem for $A \rtimes_{\alpha,L} \mathbb{N}$, which generalises the one of Exel and Vershik in [5].

Date: 22 March 2004.

This research has been supported by the Australian Research Council.

2. Relative Cuntz-Pimsner algebras

Suppose that A is a C^*-algebra and X is a Hilbert bimodule over A, where the left action $a \cdot x$ is given by a homomorphism $\phi : A \to \mathcal{L}(X)$, so that $a \cdot x = \phi(a)x$. A Toeplitz representation (ψ, π) of X in a C^*-algebra B is a pair consisting of a linear map $\psi : X \to B$ and a homomorphism $\pi : A \to B$ such that

$$\psi(x \cdot a) = \psi(x)\pi(a),\ \psi(x)^*\psi(y) = \pi(\langle x, y\rangle_A),\ \text{and}\ \psi(\phi(a)x) = \pi(a)\psi(x)$$

for $x, y \in X$ and $a \in A$. Given such a representation, [7, Proposition 1.6] says there is a homomorphism $(\psi, \pi)^{(1)} : \mathcal{K}(X) \to B$ which satisfies

$$(\psi, \pi)^{(1)}(\Theta_{x,y}) = \psi(x)\psi(y)^* \text{ for } x, y \in X,$$

and

$$(\psi, \pi)^{(1)}(T)\psi(x) = \psi(Tx) \text{ for } T \in \mathcal{K}(X) \text{ and } x \in X. \tag{2.1}$$

If $\rho : B \to C$ is a homomorphism of C^*-algebras, then $(\rho \circ \psi, \rho \circ \pi)$ is a Toeplitz representation of X, and we have

$$(\rho \circ \psi, \rho \circ \pi)^{(1)}(\Theta_{x,y}) = \rho \circ \psi(x)\rho \circ \psi(y)^* = \rho \circ (\psi, \pi)^{(1)}(\Theta_{x,y}) \text{ for all } x, y \in X.$$

It follows from linearity and continuity that we have

$$(\rho \circ \psi, \rho \circ \pi)^{(1)} = \rho \circ (\psi, \pi)^{(1)}. \tag{2.2}$$

We define

$$J(X) := \phi^{-1}(\mathcal{K}(X)),$$

which is a closed two-sided ideal in A. Let K be an ideal contained in $J(X)$. Following Muhly and Solel, we say that a Toeplitz representation (ψ, π) of X is *coisometric on K* if

$$(\psi, \pi)^{(1)}(\phi(a)) = \pi(a) \quad \text{for all} \quad a \in K.$$

Proposition 2.1. [6, Proposition 1.3] *Let X be a Hilbert bimodule over A, and let K be an ideal in $J(X)$. Then there are a C^*-algebra $\mathcal{O}(K, X)$ and a Toeplitz representation $(k_X, k_A) : X \to \mathcal{O}(K, X)$ which is coisometric on K and satisfies:*

(i) *for every Toeplitz representation (ψ, π) of X which is coisometric on K, there is a homomorphism $\psi \times_K \pi$ of $\mathcal{O}(K, X)$ such that $(\psi \times_K \pi) \circ k_X = \psi$ and $(\psi \times_K \pi) \circ k_A = \pi$; and*

(ii) *$\mathcal{O}(K, X)$ is generated as a C^*-algebra by $k_X(X) \cup k_A(A)$.*

The triple $(\mathcal{O}(K, X), k_X, k_A)$ is unique: if (B, k'_X, k'_A) has similiar properties, there is an isomorphism $\theta : \mathcal{O}(K, X) \to B$ such that $\theta \circ k_X = k'_X$ and $\theta \circ k_A = k'_A$. There is a strongly continuous gauge action $\gamma : \mathbb{T} \to \operatorname{Aut} \mathcal{O}(K, X)$ which satisfies $\gamma_z(k_A(a)) = k_A(a)$ and $\gamma_z(k_X(x)) = zk_X(x)$ for $a \in A, x \in X$.

The algebra $\mathcal{O}(K, X)$ is called the relative Cuntz-Pimsner algebra determined by K, and was first studied by Muhly and Solel in [11]. The algebra $\mathcal{O}(\{0\}, X)$ is the Toeplitz algebra $\mathcal{T}(X)$ (see [7, Proposition 1.4]), and $\mathcal{O}(J(X), X)$ is the Cuntz-Pimsner algebra $\mathcal{O}(X)$ [15]. The following lemma tells us when $k_A : A \to \mathcal{O}(K, X)$ is injective.

Lemma 2.2. *Let X be a Hilbert bimodule over A and let $(\mathcal{O}(K,X), k_A, k_X)$ be a relative Cuntz-Pimsner algebra associated to X. Then k_A is injective if and only if $\phi|_K : K \to \mathcal{L}(X)$ is injective.*

Proof. If $\phi|_K$ is injective, then [11, Proposition 2.21] implies that k_A is injective. Conversely, suppose k_A is injective and $a \in K$ satisfies $\phi|_K(a) = 0$. Then $k_A(a) = (k_X, k_A)^{(1)}(\phi|_K(a)) = 0$, and since k_A is injective, this implies $a = 0$. Thus $\phi|_K : K \to \mathcal{L}(X)$ is injective. □

3. Exel's Crossed Product

Let A be a unital C^*-algebra and α an endomorphism of A; we do not assume that α is unital or injective. In [3], Exel defined a *transfer operator* L for (A, α) to be a continuous linear map $L : A \to A$ such that

(i) L is positive in the sense that $a \geq 0 \Longrightarrow L(a) \geq 0$, and
(ii) $L(\alpha(a)b) = aL(b)$, for all $a, b \in A$.

He then defined $\mathcal{T}(A, \alpha, L)$ to be the universal unital C^*-algebra generated by a copy of A and an element S satisfying the relations $Sa = \alpha(a)S$ and $S^*aS = L(a)$ for $a \in A$, so that $\mathcal{T}(A, \alpha, L)$ is by definition universal for the following representations.

Definition 3.1. A pair (ρ, V), consisting of a unital homomorphism ρ of A into a C^*-algebra B and an element $T \in B$, is a *Toeplitz-covariant representation of* (A, α, L) in B if for every $a \in A$,

(TC1) $V\rho(a) = \rho(\alpha(a))V$, and
(TC2) $V^*\rho(a)V = \rho(L(a))$.

We denote by (i_A, S), the universal Toeplitz-covariant representation of (A, α, L) in $\mathcal{T}(A, \alpha, L)$. If (ρ, V) is a Toeplitz-covariant representation of (A, α, L), we denote by $\rho \times V$ the representation of $\mathcal{T}(A, \alpha, L)$ such that $(\rho \times V) \circ i_A = \rho$ and $(\rho \times V)(S) = V$.

The homomorphism $i_A : A \to \mathcal{T}(A, \alpha, L)$ is injective: to see this, we need an example of a Toeplitz-covariant representation (ρ, V) with ρ injective, and one such example is given in [3].

Given the triple (A, α, L), we recall from [3] the construction of the Hilbert A-bimodule M_L. We let A_L be a copy of the underlying vector space of A. We define a right action of A on A_L by

$$m \cdot a = m\alpha(a) \text{ for } m \in A_L \text{ and } a \in A,$$

and an A-valued map $\langle \cdot, \cdot \rangle_L$ on A_L by

$$\langle m, n \rangle_L = L(m^*n) \text{ for } m, n \in A_L.$$

We define $N := \{a \in A_L : \langle a, a \rangle_L = 0\}$; it follows from the Cauchy-Schwarz inequality that N is a subspace of A_L, and we can form the quotient space A_L/N. We denote the quotient map by $q : A_L \to A_L/N$, and then A_L/N is a right A-module with inner-product $\langle q(a), q(b) \rangle_L = L(a^*b)$. By completing A_L/N we get a right Hilbert A-module which we denote by M_L. For $a \in A$ and $m \in A_L$ we have

$$\|\langle am, am \rangle_L\| = \|L(m^*a^*am)\| \leq \|a\|^2 \|L(m^*m)\| = \|a\|^2 \|\langle m, m \rangle_L\|,$$

and it follows that left multiplication by a on A_L extends to a bounded adjointable operator on M_L. This defines a homomorphism $\phi : A \to \mathcal{L}(M_L)$, and writing $\phi(a)m := a \cdot m$ makes M_L a Hilbert bimodule over A. Note that $q(A_L)$ is dense in M_L.

In the following lemma we see that there is a one-to-one correspondence between Toeplitz-covariant representations of (A, α, L) and Toeplitz representations of M_L.

Lemma 3.2. *Given a Toeplitz-covariant representation (ρ, V) of (A, α, L) in a C^*-algebra B, there exists a linear map $\psi_V : M_L \to B$ such that $\psi_V(q(a)) = \rho(a)V$ and the pair (ψ_V, ρ) is a Toeplitz representation of M_L in B. Conversely, if (ψ, π) is a Toeplitz representation of M_L in B and π is unital, then the pair $(\pi, \psi(q(1)))$ is a Toeplitz-covariant representation of (A, α, L), and $\psi_{\psi(q(1))} = \psi$.*

Proof. We define $\theta : A_L \to B$ by $\theta(a) = \rho(a)V$. Then θ is linear, and for $a \in A$ we have

$$\begin{aligned}\|\theta(a)\|^2 = \|\rho(a)V\|^2 = \|(\rho(a)V)^*\rho(a)V\| = \|V^*\rho(a^*a)V\| = \|\rho(L(a^*a))\|\\ \le \|L(a^*a)\| = \|\langle a, a\rangle_L\|,\end{aligned}$$

so θ is bounded for the semi-norm on A_L. Thus θ induces a bounded map $\psi_V : M_L \to B$ satisfying $\psi_V(q(a)) = \rho(a)V$ for $a \in A$. For $a, b, c \in A$ we have

$$\psi_V(q(b) \cdot a) = \psi_V(q(b)\alpha(a)) = \rho(b\alpha(a))V = \rho(b)V\rho(a) = \psi_V(q(b))\rho(a),$$
$$\psi_V(q(b))^*\psi_V(q(c)) = (\rho(b)V)^*\rho(c)V = V^*\rho(b^*c)V = \rho(L(b^*c)) = \rho(\langle q(b), q(c)\rangle_L), \text{ and}$$
$$\psi_V(a \cdot q(b)) = \psi_V(aq(b)) = \rho(ab)V = \rho(a)\rho(b)V = \rho(a)\psi_V(q(b)).$$

Thus (ψ_V, ρ) is a Toeplitz representation of M_L in B.

Now let $(\psi, \pi) : M_L \to B$ be a Toeplitz representation of M_L ina C^*-algebra B with π unital. Then for $a \in A$ we have

$$\psi(q(1))\pi(a) = \psi(q(1) \cdot a) = \psi(q(\alpha(a)) = \psi(\alpha(a) \cdot q(1)) = \pi(\alpha(a))\psi(q(1)),$$

and

$$\begin{aligned}\psi(q(1))^*\pi(a)\psi(q(1)) &= \psi(q(1))^*\psi(a \cdot q(1)) = \psi(q(1))^*\psi(q(a))\\ &= \pi(\langle q(1), q(a)\rangle_L) = \pi(L(1^*a)) = \pi(L(a)),\end{aligned}$$

so $(\pi, \psi(q(1)))$ is a Toeplitz-covariant representation of (A, α, L). Finally, for $a \in A$ we have

$$\psi_{\psi(q(1))}(q(a)) = \pi(a)\psi(q(1)) = \psi(a \cdot q(1)) = \psi(q(a)),$$

which implies that $\psi_{\psi(q(1))} = \psi$. □

Corollary 3.3. *The C^*-algebra $\mathcal{T}(A, \alpha, L)$ is isomorphic to the Toeplitz algebra $\mathcal{T}(M_L)$.*

Proof. We prove that $\mathcal{T}(A, \alpha, L)$ has the universal property which characterises $\mathcal{T}(M_L)$. Applying the lemma to the pair (i_A, S) gives a Toeplitz representation (ψ_S, i_A) of M_L in $\mathcal{T}(A, \alpha, L)$, which generates $\mathcal{T}(A, \alpha, L)$ because i_A and S do. Now suppose (ψ, π) is a Toeplitz representation of M_L. Note that M_L is essential as a left A-module, in the sense that $A \cdot M_L = M_L$. This implies that the essential subspace $\pi(1)\mathcal{H}$ is reducing for (ψ, π), so we can apply the lemma to the restriction of (ψ, π) to $\pi(1)\mathcal{H}$; this gives

a Toeplitz-covariant representation $(\pi|, \psi|(q(1)))$ on $\pi(1)\mathcal{H}$. Now the representation $\mu := (\pi| \times \psi|(q(1))) \oplus 0$ has $\mu \circ i_A = \pi| \oplus 0 = \pi$, and for $a \in A$ we have

$$\begin{aligned}\mu \circ \psi_S(q(a)) &= \mu(i_A(a)S) = \mu(i_A(a))\mu(S) = (\pi|(a)\psi|(q(1))) \oplus 0\\ &= \pi(a)\psi(q(1)) = \psi_{\psi(q(1))}(q(a)) = \psi(q(a)),\end{aligned}$$

which implies that $\mu \circ \psi_S = \psi$. □

Corollary 3.3 has been obtained independently by Nadia Larsen.

Remark 3.4. The Toeplitz representation (ψ_S, i_A) induces a homomorphism $(\psi_S, i_A)^{(1)}$ of $\mathcal{K}(M_L)$ into $\mathcal{T}(A, \alpha, L)$. We claim that $(\psi_S, i_A)^{(1)}$ is injective. To see this, let $\pi : \mathcal{T}(A, \alpha, L) \to B(\mathcal{H})$ be a faithful non-degenerate representation of $\mathcal{T}(A, \alpha, L)$. Then, as in the proof of [7, Proposition 1.6], we have $(\psi_S, i_A)^{(1)} := \pi^{-1} \circ \operatorname{Ad} U \circ \operatorname{Ind}(\pi \circ i_A)$, where $U : M_L \otimes_A \mathcal{H} \to \mathcal{H}$ is an isometry given by $U(m \otimes_A h) = \pi(\psi_S(m))h$. Since $\pi \circ i_A$ is faithful, the induced representation is faithful [16, Corollary 2.74], and $(\psi_S, i_A)^{(1)}$ is injective, as claimed.

The range of any homomorphism of C^*-algebras is closed, and since $(\psi_S, i_A)^{(1)}(\mathcal{K}(M_L))$ is dense in $\overline{\psi_S(M_L)\psi_S(M_L)^*}$, it follows that $(\psi_S, i_A)^{(1)}$ is an isomorphism of $\mathcal{K}(M_L)$ onto the C^*-algebra $\overline{\psi_S(M_L)\psi_S(M_L)^*} = \overline{i_A(A)SS^*i_A(A)}$.

We will now discuss Exel's notion of a redundancy. Define $M := \overline{i_A(A)S} = \psi_S(M_L)$. Conditions (TC1) and (TC2) imply that $i_A(A)M \subseteq M$, $Mi_A(A) \subseteq M$ and $M^*M \subseteq i_A(A)$, so M is a Hilbert bimodule over $i_A(A)$. It follows that left multiplication by elements of $i_A(A)$ on M could coincide with left multiplication by elements in $\overline{MM^*} = \overline{i_A(A)SS^*i_A(A)}$. In [3], Exel defines a *redundancy* to be a pair $(i_A(a), k)$ such that $a \in A$, $k \in \overline{i_A(A)SS^*i_A(A)}$ and

$$i_A(a)i_A(b)S = ki_A(b)S \text{ for all } b \in A.$$

The next lemma provides a useful identification of the redundancies.

Lemma 3.5. *Let $a \in A$ and let $k \in \mathcal{T}(A, \alpha, L)$. Then $(i_A(a), k)$ is a redundancy if and only if $a \in J(M_L) := \phi^{-1}(\mathcal{K}(M_L))$ and $k = (\psi_S, i_A)^{(1)}(\phi(a))$.*

Proof. First suppose that $a \in J(M_L)$ and $k = (\psi_S, i_A)^{(1)}(\phi(a))$. Then k belongs to the image $\overline{i_A(A)SS^*i_A(A)}$ of $(\psi_S, i_A)^{(1)}$, and for $b \in A$ we have

$$\begin{aligned}i_A(a)i_A(b)S &= i_A(a)\psi_S(q(b)) = \psi_S(\phi(a)q(b))\\ &= (\psi_S, i_A)^{(1)}(\phi(a))\psi_S(q(b))\\ &= (\psi_S, i_A)^{(1)}(\phi(a))i_A(b)S,\end{aligned}$$

where the second last equality follows from Equation (2.1). Thus $(i_A(a), k)$ is a redundancy.

Now suppose that $(i_A(a), k)$ is a redundancy. It follows from Remark 3.4 that there exists a unique $t \in \mathcal{K}(M_L)$ such that $(\psi_S, i_A)^{(1)}(t) = k$. Then for $b \in A$ we have

$$\begin{aligned}\psi_S\big(\phi(a)(q(b))\big) &= \psi_S(q(ab)) = i_A(ab)S = i_A(a)i_A(b)S\\ &= ki_A(b)S = (\psi_S, i_A)^{(1)}(t)\psi_S(q(b)) = \psi_S(t(q(b))),\end{aligned}$$

Since $i_A : A \to \mathcal{T}(A, \alpha, L)$ is injective, ψ_S is also injective, and it follows that $\phi(a)(m) = t(m)$ for all $m \in M_L$. Hence $\phi(a) = t$, and the result follows. $\square$

Exel defined the crossed product of (A, α, L) to be the quotient of $\mathcal{T}(A, \alpha, L)$ by the ideal generated by the set

$$\{i_A(a) - k : (i_A(a), k) \text{ is a redundancy with } a \in \overline{A\alpha(A)A}\}.$$

We denote the quotient map by $Q : \mathcal{T}(A, \alpha, L) \to A \rtimes_{\alpha,L} \mathbb{N}$. The next corollary follows immediately from Lemma 3.5.

Corollary 3.6. *Let $K_\alpha := \overline{A\alpha(A)A} \cap J(M_L)$ and denote by $I(A, \alpha, L)$ the ideal in $\mathcal{T}(A, \alpha, L)$ generated by*

$$\{i_A(a) - (\psi_S, i_A)^{(1)}(\phi(a)) : a \in K_\alpha\}.$$

Then $A \rtimes_{\alpha,L} \mathbb{N}$ is $\mathcal{T}(A, \alpha, L)/I(A, \alpha, L)$.

To describe $A \rtimes_{\alpha,L} \mathbb{N}$ as a universal object, we need to identify the Toeplitz-covariant representations that vanish on the ideal $I(A, \alpha, L)$. We need a lemma:

Lemma 3.7. *Suppose (ρ, V) is a covariant representation of (A, α, L). Then we have*

$$(\rho \times V) \circ (\psi_S, i_A)^{(1)} = (\psi_V, \rho)^{(1)}. \tag{3.1}$$

Proof. We know from (2.2) that

$$(\rho \times V) \circ (\psi_S, i_A)^{(1)} = ((\rho \times V) \circ \psi_S, (\rho \times V) \circ i_A)^{(1)}.$$

Since (i_A, S) is the universal Toeplitz-covariant representation, we have $(\rho \times V) \circ i_A = \rho$, and $(\rho \times V)(S) = V$. So for $a \in A$ we have

$$(\rho \times V) \circ \psi_S(q(a)) = \rho \times V(i_A(a)S) = \rho(a)V = \psi_V(q(a)),$$

and hence we also have $(\rho \times V) \circ \psi_S = \psi_V$. $\square$

Equation (3.1) motivates the following definition.

Definition 3.8. Consider the triple (A, α, L), and let (ρ, V) be a Toeplitz-covariant representation in a C^*-algebra B. We say that (ρ, V) is a *covariant representation* of (A, α, L) if in addition we have

(C3) $\rho(a) = (\psi_V, \rho)^{(1)}(\phi(a))$ for all $a \in K_\alpha$.

The following Proposition says that $A \rtimes_{\alpha,L} \mathbb{N}$ is universal for covariant representations of (A, α, L).

Proposition 3.9. *Let α be an endomorphism of a unital C^*-algbera A, and let L be a transfer operator for (A, α). The pair $(j_A, T) := (Q \circ i_A, Q(S))$ is a covariant representation of (A, α, L) in $A \rtimes_{\alpha,L} \mathbb{N}$, and for every covariant representation (ρ, V) of (A, α, L), there is a representation $\tau_{\rho,V}$ of $A \rtimes_{\alpha,L} \mathbb{N}$ such that $\tau_{\rho,V} \circ j_A = \rho$ and $\tau_{\rho,V}(T) = V$.*

Proof. The pair $(Q\circ i_A, Q(S))$ is Toeplitz-covariant because (i_A, S) is, and its integrated form $(Q\circ i_A)\times Q(S)$ is precisely Q. By Lemma 3.2, we get a Toeplitz representation $(\psi_{Q(S)}, Q\circ i_A) : M_L \to A\rtimes_{\alpha,L}\mathbb{N}$, and for $a \in K_\alpha$ we have

$$\begin{aligned}(Q\circ i_A)(a) = Q(i_A(a)) &= Q\big((\psi_S, i_A)^{(1)}(\phi(a))\big)\\ &= ((Q\circ i_A)\times Q(S))\big((\psi_S, i_A)^{(1)}(\phi(a))\big)\\ &= (\psi_{Q(S)}, Q\circ i_A)^{(1)}(\phi(a)),\end{aligned}$$

using Lemma 3.7. So the pair $(Q\circ i_A, Q(S))$ is covariant.

Now suppose (ρ, V) is a covariant representation of (A,α,L). The Toeplitz-covariant representation (ρ, V) gives us a representation $\rho\times V$ of $\mathcal{T}(A,\alpha,L)$, and condition (C3) says that $\rho\times V$ vanishes on the generators of the ideal $I(A,\alpha,L)$. Hence Corollary 3.6 implies that $\rho\times V$ factors through a representation $\tau_{\rho,V}$ of $A\times_{\alpha,L}\mathbb{N}$. Then

$$\tau_{\rho,V}\circ j_A = \tau_{\rho,V}\circ Q\circ i_A = (\rho\times V)\circ i_A = \rho, \text{ and}$$
$$\tau_{\rho,V}(T) = \tau_{\rho,V}(Q(T)) = (\rho\times V)(T) = V,$$

so $\tau_{\rho,V}$ has the required properties. □

We now realise $A\rtimes_{\alpha,L}\mathbb{N}$ as a relative Cuntz-Pimsner algebra.

Proposition 3.10. *Suppose α is an endomorphism of a unital C^*-algebra A and L is a transfer operator for (A,α). Then there is an isomorphism $\theta : \mathcal{O}(K_\alpha, M_L)\to A\rtimes_{\alpha,L}\mathbb{N}$ such that $\theta\circ k_A = j_A$ and $\theta(k_{M_L}(q(1))) = T$.*

Proof. Consider the triple (A, ψ_T, j_A), where (ψ_T, j_A) is the Toeplitz representation of M_L induced by the pair (j_A, T), as in Lemma 3.2. We will prove that $(A\rtimes_{\alpha,L}\mathbb{N}, \psi_T, j_A)$ satisfies the conditions of Proposition 2.1.

Since (j_A, T) is covariant, it satisfies (C3), which says precisely that (ψ_T, j_A) is coisometric on K_α. Let (ψ,π) be a Toeplitz representation of M_L which is coisometric on K_α; since M_L is essential, we suppose by throwing away a trivial representation that π is unital (see the proof of Corollary 3.3). Then Lemma 3.2 gives a Toeplitz-covariant representation $(\pi, \psi(q(1)))$. Since $\psi_{\psi(q(1))} = \psi$ and (ψ,π) is coisometric on K_α, $(\pi,\psi(q(1)))$ is covariant. Now Proposition 3.9 gives a representation $\tau_{\pi,\psi(q(1))}$ of $A\rtimes_{\alpha,L}\mathbb{N}$ such that $\tau_{\pi,\psi(q(1))}\circ j_A = \pi$ and $\tau_{\pi,\psi(q(1))}(T) = \psi(q(1))$. For $a\in A$ we have

$$\tau_{\pi,\psi(q(1))}(\psi_T(q(a))) = \tau_{\pi,\psi(q(1))}(j_A(a)T) = \pi(a)\psi(q(1)) = \psi(q(a)),$$

and it follows that $\tau_{\pi,\psi(q(1))}\circ\psi_T = \psi$. So $\psi\times_{K_\alpha}\pi := \tau_{\pi,\psi(q(1))}$ satisfies condition (i) of Proposition 2.1. Since $\psi_T(M_L)\cup j_A(A)$ generates $A\rtimes_{\alpha,L}\mathbb{N}$, condition (ii) is also satisfied, and applying Proposition 2.1 gives the result. □

Notice that when $\alpha(1) = 1$, we have $K_\alpha = J(M_L)$, and the crossed product $A\rtimes_{\alpha,L}\mathbb{N}$ is the Cuntz-Pimsner algebra $\mathcal{O}(M_L)$.

4. Injectivity of $j_A : A \to A\rtimes_{\alpha,L}\mathbb{N}$

Definition 4.1. Suppose that A is a unital C^*-algebra, α is an endomorphism of A and L is a transfer operator for (A,α). We say that L is *faithful* on an ideal I of A if

$$a\in I \text{ and } L(a^*a) = 0 \Longrightarrow a = 0;$$

we say that L is *almost faithful* on I if

$$a \in I \text{ and } L((ab)^*ab) = 0 \text{ for all } b \in A \Longrightarrow a = 0.$$

Theorem 4.2. *Let α be an endomorphism of a unital C^*-algebra A, and let L be a transfer operator for (A, α). Then the map $j_A : A \to A \rtimes_{\alpha,L} \mathbb{N}$ is injective if and only if L is almost faithful on $K_\alpha = \overline{A\alpha(A)A} \cap J(M_L)$.*

Proof. It follows from Proposition 3.10 that the map j_A is injective if and only if $k_A : A \to \mathcal{O}(K_\alpha, M_L)$ is injective. By Lemma 2.2 this is true if and only if $\phi|_{K_\alpha} : K_\alpha \to \mathcal{L}(M_L)$ is injective, and so it suffices to prove that the transfer operator L is almost faithful on K_α if and only if $\phi|_{K_\alpha} : K_\alpha \to \mathcal{L}(M_L)$ is injective. But for $a \in K_\alpha$ and $b \in A$, we have

$$\begin{aligned} \|L((ab)^*ab)\| &= \|\langle q(ab), q(ab)\rangle_L\| = \|q(ab)\|^2 = \|a \cdot q(b)\|^2 \\ &= \|\phi(a)(q(b))\|^2 = \|\phi|_{K_\alpha}(a)(q(b))\|^2, \end{aligned}$$

and this implies the desired equivalence. □

Corollary 4.3. *Let α be an endomorphism of a unital commutative C^*-algebra A, and let L be a transfer operator for (A, α). Then the map $j_A : A \to A \rtimes_{\alpha,L} \mathbb{N}$ is injective if and only if L is faithful on K_α.*

Proof. If L is faithful on K_α then it follows from Theorem 4.2 that $j_A : A \to A \rtimes_{\alpha,L} \mathbb{N}$ is injective. Conversely, suppose $j_A : A \to A \rtimes_{\alpha,L} \mathbb{N}$ is injective. By Theorem 4.2, this implies that L is almost faithful on K_α. Suppose $a \in K_\alpha$ satisfies $L(a^*a) = 0$. Then for every $b \in A$ we have

$$\|L((ab)^*ab)\| = \|L((ba)^*ba)\| = \|L(a^*b^*ba)\| \leq \|b\|^2 \|L(a^*a)\| = 0.$$

Thus $L((ab)^*ab) = 0$ for every $b \in A$, which implies $a = 0$, and we have shown that L is faithful on K_α. □

In [4], Exel assumed that α is a unital injective endomorphism and $L = \alpha^{-1} \circ E$, where E is a conditional expectation of A onto $\alpha(A)$ satisfying $E(a^*a) = 0 \Longrightarrow a = 0$ (Exel says E is non-degenerate). Under these conditions he proves that $j_A : A \to A \rtimes_{\alpha,L} \mathbb{N}$ is injective [4, Theorem 4.12]. Notice that such L are faithful, and so [4, Theorem 4.12] follows from Theorem 4.2. The following examples show that our theorem is stronger in several different ways.

Example 4.4. In this example, the endomorphism is not unital. Let $A = c$, the space of convergent sequences under the sup norm, and let α be the forward shift τ_f. Then the backward shift $L = \tau_b$ is a transfer operator for (c, τ_f) and we have

$$M_{\tau_b} = c/\mathbb{C}e_0, \ J(M_{\tau_b}) = c, \text{ and } K_{\tau_f} = \{f \in c : f(0) = 0\};$$

notice that $L = \tau_b$ is faithful on K_{τ_f}, but not on all of c. It follows from Corollary 4.3 that the map $j_c : c \to c \rtimes_{\tau_f, \tau_b} \mathbb{N}$ is injective.

Example 4.5. In this example, the endomorphism is not injective. Again the algebra A is c, but now we view the backward shift τ_b as the endomorphism, and take for L the forward shift τ_f. Then we have $M_{\tau_f} = A_L$, $J(M_{\tau_f}) = c$, and $K_{\tau_b} = c$. In this case, $L = \tau_f$ is faithful on K_{τ_b}, so Corollary 4.3 shows that $j_c : c \to c \rtimes_{\tau_b, \tau_f} \mathbb{N}$ is injective.

Example 4.6. In this example, the transfer operator is almost faithful but is not faithful. We take A to be the UHF algebra $\mathrm{UHF}(n^\infty)$, viewed as the direct limit $\varinjlim(A_N, i_N)$ with $A_N = \bigotimes_{k=1}^N M_n(\mathbb{C})$ and

$$i_N(a_1 \otimes \cdots \otimes a_N) := a_1 \otimes \cdots \otimes a_N \otimes 1;$$

we denote the canonical embeddings by $i^N : A_N \to A$. The maps $\alpha_N : A_N \to A_{N+1}$ defined by

$$\alpha_N(a_1 \otimes \cdots \otimes a_N) = e_{11} \otimes a_1 \otimes \cdots \otimes a_N,$$

induce an injective endomorphism $\alpha : A \to A$ such that $\alpha(i^N(a)) = i^{N+1}(\alpha_N(a))$ for $a \in A_N$. Since $\operatorname{range} \alpha$ is closed, it follows that $\operatorname{range} \alpha = i^1(e_{11})A\, i^1(e_{11})$. We can then define $L : A \to A$ by

$$L(a) = \alpha^{-1}(i^1(e_{11})a\, i^1(e_{11})).$$

Then L is positive, continuous and linear. To see that L is a transfer operator, let $a = \bigotimes a_i \in A_N$, $b = \bigotimes b_i \in A_{N+1}$, and compute:

$$\begin{aligned}
L(\alpha(i^N(a))i^{N+1}(b)) &= L(i^{N+1}(e_{11}b_1 \otimes a_1b_2 \otimes \cdots \otimes a_Nb_{N+1})) \\
&= \alpha^{-1}(i^1(e_{11})i^{N+1}(e_{11}b_1 \otimes a_1b_2 \otimes \cdots \otimes a_Nb_{N+1})i^1(e_{11})) \\
&= (b_1)_{11}\alpha^{-1}(i^{N+1}(e_{11} \otimes a_1b_2 \otimes \cdots \otimes a_Nb_{N+1})) \\
&= (b_1)_{11}i^N(a_1b_2 \otimes \cdots \otimes a_Nb_{N+1}) \\
&= (b_1)_{11}i^N(a_1 \otimes \cdots \otimes a_N)i^N(b_2 \otimes \cdots \otimes b_{N+1}) \\
&= i^N(a)(b_1)_{11}\alpha^{-1}(i^{N+1}(e_{11} \otimes b_2 \otimes \cdots \otimes b_M)) \\
&= i^N(a)\alpha^{-1}(i^1(e_{11})i^{N+1}(b)i^1(e_{11})) \\
&= i^N(a)L(i^{N+1}(b)).
\end{aligned}$$

It follows from linearity and continuity of L and α that $L(\alpha(a)b) = aL(b)$ for all $a, b \in A$, and hence L is a transfer operator for (A, α).

For $j \in \{1, \ldots, n\}$ define $b_j := i^1(e_{j1})$. Suppose $a \in A$ satisfies $L((ab)^*ab) = 0$ for all $b \in A$. Then $0 = L((ab_j)^*ab_j) = \alpha^{-1}(i^1(e_{11}){b_j}^*a^*ab_ji^1(e_{11}))$ for all j, and this implies that $ab_j = 0$ for all j. Thus

$$0 = \sum_{j=1}^n ab_j{b_j}^* = ai^1\Big(\sum_{j=1}^n e_{jj}\Big) = ai^1(1) = a,$$

and hence L is almost faithful on A. To see that L is not faithful we let $a_0 \in M_n(\mathbb{C})$ be a non-zero matrix whose first column is zero. Then $({a_0}^*a_0)_{11} = 0$ and

$$L(i^1(a_0)^*i^1(a_0)) = \alpha^{-1}(i^1(e_{11}{a_0}^*a_0e_{11})) = \alpha^{-1}(({a_0}^*a_0)_{11}i^1(e_{11})) = \alpha^{-1}(0) = 0,$$

whereas $i^1(a_0) \neq 0$ because i^1 is injective.

The endomorphism α is injective and has hereditary range. Under these assumptions, Exel proved in [3, Theorem 4.7] that $A \rtimes_{\alpha,L} \mathbb{N}$ is isomorphic to the Stacey crossed product $A \rtimes_\alpha \mathbb{N}$. This crossed product was first considered by Cuntz, who showed in [2] that $\mathrm{UHF}(n^\infty) \rtimes_\alpha \mathbb{N}$ is isomorphic to the Cuntz algebra $\mathcal{O}_n$.

Example 4.7. This is an example of a commutative C^*-algebra with a transfer operator L which is not faithful on K_α, so that A does not embed in Exel's crossed product. Let $A := C([0,2])$, and define $\alpha : C([0,2]) \to C([0,2])$ by

$$\alpha(f)(x) := \begin{cases} f(2x) & \text{if } x \in [0,1] \\ f(4-2x) & \text{if } x \in (1,2]. \end{cases}$$

Then the map $L : C([0,2]) \to C([0,2])$ defined by $L(f)(x) = f(x/2)$, is a transfer operator for (A,α). We have $A_L = C([0,2])$ as a vector space, and

$$\begin{aligned} N &:= \{f \in C([0,2]) : L(f^*f) = 0\} \\ &= \{f \in C([0,2]) : f(x) = 0 \text{ for all } x \in [0,1]\}. \end{aligned}$$

Thus the restriction map $r : f \mapsto f|_{[0,1]}$ induces a vector-space isomorphism of A_L/N onto $C([0,1])$, which converts the bimodule structure into

$$\langle g,h\rangle_L(x) = \overline{g(x/2)}h(x/2),\ g\cdot f(x) = g(x)f(2x),\ f\cdot g(x) = f(x)g(x)$$

for $g,h \in C([0,2])$ and $f \in A = C([0,2])$; it follows from the first formula that r is isometric for the sup-norm on $C([0,1])$, so A_L/N is complete and $M_L = A_L/N$. Now for $f \in A$ and $x \in [0,1]$, we have

$$\Theta_{r(f),1}(g)(x) = r(f)(x)\langle 1,g\rangle_L(2x) = f(x)g(x) = (\phi(f)g)(x),$$

so $f \in J(M_L)$. Thus $J(M_L) = A$, which implies $K_\alpha = A$ because $\alpha(1) = 1$. The transfer function L is not faithful on $C([0,2])$: any nonzero function $f \in C([0,2])$ with $f|_{[0,1]} = 0$ will satisfy $L(f^*f) = 0$. Hence it follows from Corollary 4.3 that the canonical map $C([0,2]) \to C([0,2])\rtimes_{\alpha,L}\mathbb{N}$ is not injective.

5. Gauge Invariant Uniqueness Theorem

Using the isomorphism $\theta : \mathcal{O}(K_\alpha, M_L) \to A\rtimes_{\alpha,L}\mathbb{N}$ of Proposition 3.10, we can see that there is a natural gauge action $\delta : \mathbb{T} \to \operatorname{Aut}(A\rtimes_{\alpha,L}\mathbb{N})$ such that $\delta_z(j_A(a)) = j_A(a)$, $\delta_z(T) = zT$ and $\theta\circ\gamma_z = \delta_z\circ\theta$.

Theorem 5.1. *Let α be an endomorphism of a unital C^*-algebra A, and let L be a transfer operator for (A,α). Suppose B is a C^*-algebra and (ρ,V) is a covariant representation of (A,α,L) in B satisfying*

1. *for $a \in A$, $\rho(a) = 0 \Longrightarrow j_A(a) = 0$,*
2. *if $\rho(a) \in (\psi_V,\rho)^{(1)}(\mathcal{K}(M_L))$, then $j_A(a) \in j_A(K_\alpha)$,*
3. *there exists a strongly continuous action $\beta : \mathbb{T} \to \operatorname{Aut}\tau_{\rho,V}(A\rtimes_{\alpha,L}\mathbb{N})$ such that $\beta_z\circ\tau_{\rho,V} = \tau_{\rho,V}\circ\delta_z$ for all $z \in \mathbb{T}$.*

Then the corresponding representation $\tau_{\rho,V} : A\rtimes_{\alpha,L}\mathbb{N} \to B$ is faithful.

The proof of Theorem 5.1 will use the following gauge-invariant uniqueness theorem for relative Cuntz-Pimsner algebras, which is due to Katsura [8, Corollary 11.7] and Muhly-Tomforde [12, §5].

Theorem 5.2. *Suppose X is a Hilbert bimodule over A and K is an ideal in $J(M_L)$. If $\mu : \mathcal{O}(K,X) \to B$ is a homomorphism into a C^*-algbera B satisfying*

(i) *the restriction of μ to $k_A(A)$ is injective,*

(ii) *if* $\mu(k_A(a)) \in \mu\big((k_X, k_A)^{(1)}(\mathcal{K}(X))\big)$*, then* $k_A(a) \in k_A(K)$*,*
(iii) *there exists a strongly continuous action* $\beta : \mathbb{T} \to \operatorname{Aut}\mu(\mathcal{O}(K, X))$ *such that* $\beta_z \circ \mu = \mu \circ \gamma_z$ *for all* $z \in \mathbb{T}$*,*

then μ *is injective.*

Proof of Theorem 5.1. We will prove that $\tau_{\rho,V} \circ \theta$ satisfies the conditions of Theorem 5.2. Suppose $a \in A$ satisfies $(\tau_{\rho,V} \circ \theta)(k_A(a)) = 0$. Then

$$\rho(a) = \tau_{\rho,V}(j_A(a)) = \tau_{\rho,V}(\theta(a)) = 0,$$

which by (1) implies that $j_A(a) = 0$. Hence $k_A(a) = \theta^{-1}(j_A(a)) = 0$, and so $\tau_{\rho,V} \circ \theta$ is injective on $k_A(A)$.

Now suppose $a \in A$ and $(\tau_{\rho,V} \circ \theta)(k_A(a)) \in (\tau_{\rho,V} \circ \theta)\big((k_{M_L}, k_A)^{(1)}(\mathcal{K}(M_L))\big)$. We have $(\tau_{\rho,V} \circ \theta)(k_A(a)) = \rho(a)$, and Lemma 3.7 gives

$$\begin{aligned}(\tau_{\rho,V} \circ \theta)\big((k_{M_L}, k_A)^{(1)}(\mathcal{K}(M_L))\big) &= \tau_{\rho,V}\big((\theta \circ k_{M_L}, \theta \circ k_A)^{(1)}(\mathcal{K}(M_L))\big) \\ &= \tau_{\rho,V}\big((\psi_T, j_A)^{(1)}(\mathcal{K}(M_L))\big) \\ &= \tau_{\rho,V} \circ Q\big((\psi_S, i_A)^{(1)}(\mathcal{K}(M_L))\big) \\ &= (\rho \times V)\big((\psi_S, i_A)^{(1)}(\mathcal{K}(M_L))\big) \\ &= (\psi_V, \rho)^{(1)}(\mathcal{K}(M_L)).\end{aligned}$$

So $\rho(a) \in (\psi_V, \rho)^{(1)}(\mathcal{K}(M_L))$, and then it follows from (2) that $j_A(a) \in j_A(K_\alpha)$. Hence $k_A(a) \in k_A(K_\alpha)$. By (3), we have

$$\beta_z \circ \tau_{\rho,V} \circ \theta = \tau_{\rho,V} \circ \delta_z \circ \theta = \tau_{\rho,V} \circ \theta \circ \gamma_z,$$

so Theorem 5.2 implies that $\tau_{\rho,V} \circ \theta$ is injective. Thus $\tau_{\rho,V}$ is injective. □

When the transfer operator L is almost faithful on K_α, our main theorem says that j_A is injective. Using [8, Corollary 11.8] instead of Theorem 5.2 yields the following gauge-invariant uniqueness theorem which directly generalises [5, Theorem 4.2] (because the second condition (2′) trivially holds when $K_\alpha = J(M_L)$, as is the case when $\alpha(1) = 1$).

Corollary 5.3. *Let* α *be an endomorphism of a unital* C^**-algebra* A*, and let* L *be a transfer operator for* (A, α) *which is almost faithful on* K_α*. Suppose* B *is a* C^**-algebra and* (ρ, V) *is a covariant representation of* (A, α, L) *in* B *satisfying*

(1′) ρ *is faithful,*
(2′) *for* $a \in J(M_L)$*,* $\rho(a) = (\psi_V, \rho)^{(1)}(\phi(a))$ *implies* $j_A(a) = (\psi_T, j_A)^{(1)}(\phi(a))$*,*
(3) *there exists a strongly continuous action* $\beta : \mathbb{T} \to \operatorname{Aut}\tau_{\rho,V}(A \rtimes_{\alpha,L} \mathbb{N})$ *such that* $\beta_z \circ \tau_{\rho,V} = \tau_{\rho,V} \circ \gamma_z$ *for all* $z \in \mathbb{T}$*.*

Then the corresponding representation $\tau_{\rho,V} : A \rtimes_{\alpha,L} \mathbb{N} \to B$ *is faithful.*

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